6. Integracija trigonometrijskih f-ja

Dosta često se pojavljuju integrali izraza, koji sadrže trigonometrijske f-je slijedećih tipova:

I. $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ II. $\int \sin^m x \cos^n x\,dx$

III. $\int tg^n x\,dx$, $\int ctg^n x\,dx$

gdje su m i n - pozitivni cijeli brojevi,

IV. $\int \sin ax \cos bx\,dx$, $\int \sin ax \sin bx\,dx$,

$\int \cos ax \cos bx\,dx$

koji se mogu svesti na formulu integriranja, a prematome i naći, tako što će se slijèditi neko od pravila:

1. Integrali od parnog stepena sinusa ili kosinusa mogu se odrediti pomoću smanjivanja stepena (dvaput) pomoću formula:

$\sin^2 u = \frac{1}{2}(1-\cos 2u)$; $\cos^2 u = \frac{1}{2}(1+\cos 2u)$; $\sin u \cos u = \frac{1}{2}\sin 2u$.

2. Integrale neparnog stepena od sinusa ili kosinusa možemo odrediti putem razdvajanja jednog od drugog faktora i zameniti komplementarnu f-ju

novom promjenjivom.

3. Integrale tipa II možemo odrediti po pravilu 1, ako su oba broja m i n parna, ili po pravilu 2 ako su m ili n (ili oba) neparna.

4. Integrale tipa III možemo odrediti putem zamjene tg x, ili dosljedno, ctg x novom promjenjivom.

5. Integrale tipa IV možemo odrediti tako što ćemo razložiti podintegralni izraz na dijelove pomoću formula

$$\sin ax \cos bx = \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right]$$

$$\sin ax \sin bx = \frac{1}{2}\left[\cos(a-b)x - \cos(a+b)x\right]$$

$$\cos ax \cos bx = \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right]$$

(#) Odrediti integrale

a) $\int \sin^2 3x\,dx$; b) $\int \cos^4 x\,dx$; c) $\int \sin^5 x\,dx$;

d) $\int \sin^4 x \cos^2 x\,dx$; e) $\int \sin^6 kx \cos^3 kx\,dx$; f) $\int \sin^3 x \cos^5 x\,dx$.

Rj.

a) Prema pravilu 1 imamo

$$\int \sin^2 3x\,dx = \frac{1}{2}\int (1-\cos 6x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\cdot\frac{1}{6}\int \cos 6x\,d(6x) =$$
$$= \frac{x}{2} - \frac{1}{12}\sin 6x + C.$$

b) Prema pravilu 1 imamo

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx =$$
$$= \frac{1}{4}\left[\int dx + 2\int \cos 2x \cdot \frac{1}{2}\,d(2x) + \underbrace{\int \cos^2 2x\,dx}_{I_1}\right]$$

$$I_1 = \int \cos^2 2x\,dx = \frac{1}{2}\int (1+\cos 4x)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\cdot\frac{1}{4}\int \cos 4x\,d(4x) =$$
$$= \frac{1}{2}x + \frac{1}{8}\sin 4x$$

Prema tome

$$\int \cos^4 x\,dx = \frac{1}{4}\left(x + \sin 2x + \frac{x}{2} + \frac{1}{8}\sin 4x\right) + C$$
$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

c) $\int \sin^5 x\,dx = \left| \begin{array}{l} \text{prema pravilu 2} \\ \sin^5 x = \sin^4 x \cdot \sin x \end{array} \right| = \int (\sin^2 x)^2 \sin x\,dx =$

$= \int (1-\cos^2 x)^2 \sin x\,dx = \left| \begin{array}{l} \cos x = z \\ -\sin x\,dx = dz \\ \sin x\,dx = -dz \end{array} \right| = \int (1-z^2)^2(-dz) =$

$= -\int (1-2z^2+z^4)\,dz = -z + \frac{2}{3} z^3 - \frac{1}{5} z^5 + C =$

$= C - \cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$

d) $\int \sin^4 x \cos^2 x\,dx = \left| \begin{array}{l} \text{prema pravilu 3} \\ \sin u \cos u = \frac{1}{2}\sin 2u \\ \sin^2 = \frac{1}{2}(1-\cos 2\ ) \end{array} \right| =$

$= \int \sin^2 x \sin^2 x \cos^2 x\,dx = \int \sin^2 x (\sin x \cos x)^2\,dx =$

$= \int \frac{1}{2}(1-\cos 2x)\cdot \frac{1}{4}\sin^2 2x\,dx = \frac{1}{8}\underbrace{\int \sin^2 2x\,dx}_{I_1} -$

$- \frac{1}{8}\underbrace{\int \sin^2 2x \cos 2x\,dx}_{I_2} = \frac{1}{8}(I_1 - I_2)$

$I_1 = \int \sin^2 2x\,dx = \frac{1}{2}\int (1-\cos 4x)\,dx = \frac{1}{2}\int dx - \frac{1}{8}\int \cos 4x\,d(4x)$

$= \frac{1}{2}x - \frac{1}{8}\sin 4x$

$I_2 = \int \sin^2 2x \cos 2x\,dx = \left| \begin{array}{l} \sin 2x = z \\ 2\cos 2x\,dx = dz \end{array} \right| = \frac{1}{2}\int z^2\,dz = \frac{1}{2}\cdot\frac{z^3}{3}$

$= \frac{1}{6}\sin^3 2x$.

Prema tome

$$\int \sin^4 x \cos^2 x \, dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{6}\sin^3 2x\right) + C$$

e) $$\int \sin^6 kx \cos^3 kx \, dx = \left| \begin{array}{l} \text{prema pravilu 3 pišemo} \\ \cos^3 kx = \cos^2 kx \cdot \cos kx \\ = (1-\sin^2 kx)\cos kx \end{array} \right| =$$

$$= \int \sin^6 kx \,(1-\sin^2 kx) \cos kx \, dx = \left| \begin{array}{l} \sin kx = z \\ k\cos kx \, dx = dz \end{array} \right| =$$

$$= \frac{1}{k}\int z^6 (1-z^2)\, dz = \frac{1}{k}\left(\int z^6 dz - \int z^8 dz\right) =$$

$$= \frac{1}{k}\left(\frac{z^7}{7} - \frac{z^9}{9}\right) + C = \frac{1}{7k}\sin^7 kx - \frac{1}{9k}\sin^9 kx + C$$

f) $$\int \sin^3 x \cos^5 x \, dx = \left| \begin{array}{l} \text{prema pravilu 3} \\ \sin^3 x = \sin^2 x \sin x \\ = (1-\cos^2 x)\sin x \end{array} \right| =$$

$$= \int (1-\cos^2 x)\cos^5 x \sin x \, dx = \left| \begin{array}{l} \cos x = z \\ -\sin x \, dx = dz \end{array} \right| =$$

$$= -\int (1-z^2) z^5 dz = -\int z^5 dz + \int z^7 dz = \frac{1}{8}z^8 - \frac{1}{6}z^6 + C$$

$$= \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C.$$

Odrediti integrale

a) $\int tg^4 x \, dx$ b) $\int \sin 3x \cos 5x \, dx$

Rj:

a) $\int tg^4 x \, dx = \left| \begin{array}{l} \text{primjenom pravila } 4 \\ tg\, x = z \\ x = arc\, tg\, z \\ dx = \frac{dz}{1+z^2} \end{array} \right| = \int \frac{z^4}{z^2+1} dz \overset{(*)}{=}$

$$z^4 : (z^2+1) = z^2 - 1$$
$$\underline{-\; z^4 + z^2}$$
$$-z^2$$
$$\underline{-\; -z^2 - 1}$$
$$1$$

$$\frac{z^4}{z^2+1} = z^2 - 1 + \frac{1}{z^2+1}$$

$$\overset{(*)}{=} \int \left(z^2 - 1 + \frac{1}{z^2+1} \right) dz = \frac{1}{3} z^3 - z + arc\, tg\, z + C =$$

$$= \frac{1}{3} tg^3 x - tg\, x + x + C.$$

b)

$$\int \sin 3x \cos 5x \, dx = \left| \begin{array}{l} \text{primjenom pravila } 5 \\ \sin 3x \cos 5x = \frac{1}{2}\left[\sin 8x + \sin(-2x)\right] \end{array} \right| =$$

$$= \frac{1}{2} \int \left[\sin 8x - \sin 2x \right] dx = \frac{1}{2} \cdot \frac{1}{8} \int \sin 8x \, d(8x) -$$

$$- \frac{1}{2} \cdot \frac{1}{2} \int \sin 2x \, d(2x) = -\frac{1}{16} \cos 8x + \frac{1}{4} \cos 2x + C$$

Zadaci za vježbu

Odrediti integrale

1. $\int \cos^2 5x\,dx$ 2. $\int \cos^5 x\,dx$ 3. $\int \sin^2 x \cos^2 x\,dx$

4. $\int \sin^3 x \cos^2 x\,dx$ 5. $\int \sin^3 x \cos^3 x\,dx$ 6. $\int \sin^4 x\,dx$

7. $\int \mathrm{ctg}^4 y\,dy$ 8. $\int \cos\frac{4}{3}x \cos 3x\,dx$ 9. $\int \sin 5x \sin 6x\,dx$

10. $\int \sin at \cos bt\,dt$ 11.* $\int \sin 3x \sin 4x \sin 5x\,dx$

12.* $\int (\mathrm{tg}\,z + \mathrm{ctg}\,z)^3\,dz$

Rješenja:

1. $\frac{x}{2} + \frac{1}{20}\sin 10x$ 2. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x$ 3. $\frac{x}{8} - \frac{1}{32}\sin 4x$

4. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$ 5. $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x$ 6. $\frac{3}{8}x - \frac{1}{4}\sin 2x +$
$+ \frac{1}{32}\sin 4x$ 7. $y + \mathrm{ctg}\,y - \frac{1}{3}\mathrm{ctg}^3 y$ 8. $\frac{3}{26}\sin\frac{13}{3}x + \frac{3}{10}\sin\frac{5}{3}x$

9. $\frac{1}{2}\sin x - \frac{1}{22}\sin 11x$ 10. $\frac{\cos(a-b)t}{2(b-a)} - \frac{\cos(a+b)t}{2(a+b)}$

11. $\frac{\cos 12x}{48} - \frac{\cos 6x}{24} - \frac{\cos 4x}{16} - \frac{\cos 2x}{8}$

12. $\frac{1}{2}(\mathrm{tg}^2 z - \mathrm{ctg}^2 z) + 2\ln|\mathrm{tg}\,z|.$

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Integracija trigonometriskih funkcija)

$$\int \sin^m x \cdot \cos^n x \, dx \quad (m, n \in \mathbb{N}_0)$$

ako je m neparan uvodimo smjenu $\cos x = t$
ako je n neparan uvodimo smjenu $\sin x = t$
ako su m i n parni koristimo formule

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

1. $$\int \sin^3 x \cdot \cos^{12} x \, dx = \int \sin x \cdot \sin^2 x \cdot \cos^{12} x \, dx = \int \underline{\sin x} (1-\cos^2 x) \cos^{12} x \, \underline{dx}$$
$$= \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ \sin x \, dx = -dt \end{array} \right| = \int (1-t^2) \cdot t^{12} \cdot (-dt) = -\int (t^{12} - t^{14}) \, dt = -\int t^{12} dt +$$
$$+ \int t^{14} dt = -\frac{t^{13}}{13} + \frac{t^{15}}{15} + C = -\frac{1}{13} \cos^{13} x + \frac{1}{15} \cos^{15} x + C$$

2. $$\int \sin^4 x \cdot \cos^5 x \, dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx$$
$$= \int \sin^4 x (1-\sin^2 x)^2 \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right| = \int t^4 \cdot (1-t^2)^2 \, dt =$$
$$= \int t^4 (1 - 2t^2 + t^4) \, dt = \int (t^8 - 2t^6 + t^4) \, dt = \frac{t^9}{9} - 2 \cdot \frac{t^7}{7} + \frac{t^5}{5} + C$$
$$= \frac{1}{9} \sin^9 x - \frac{2}{7} \sin^7 x + \frac{1}{5} \sin^5 x + C$$

3. ✓ $$\int \sin^7 x \cdot \cos^{10} x \, dx$$

4. ✓ $$\int \sin^2 x \cdot \cos^9 x \, dx$$

5. ✓ $$\int \sin^5 x \, dx$$

Rj: $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

6. $$\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx =$$

$$=\int \frac{1+2\cos 2x+\cos^2 2x}{4}dx=\frac{1}{4}\int dx+\frac{2}{4}\int \cos 2x\,dx+\frac{1}{4}\int \cos^2 2x\,dx=$$

$$=\frac{1}{4}x+\frac{1}{2}\cdot\frac{1}{2}\sin 2x+\frac{1}{4}\int \frac{1+\cos 4x}{2}dx=\frac{1}{4}x+\frac{1}{4}\sin 2x+\frac{1}{8}\left(\int dx+\int \cos 4x\,dx\right)$$

$$\int \cos 2x\,dx=\left|\begin{matrix}2x=t\\ 2dx=dt\\ dx=\frac{dt}{2}\end{matrix}\right|=\int \cos t\cdot\frac{dt}{2}=\frac{1}{2}\sin t+C=\frac{1}{2}\sin 2x+C$$

$$=\frac{1}{4}x+\frac{1}{4}\sin 2x+\frac{1}{8}x+\frac{1}{8}\cdot\frac{1}{4}\sin 4x+C=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

Napomena: Zadatak možemo riješiti i parcijalnom integracijom $\int \cos^4 x\,dx=\int \cos^3 x\cdot\cos x\,dx=\left|\begin{matrix}u=\cos^3 x & dv=\cos x\,dx\\ \vdots & \end{matrix}\right|$

7. $$\int \sin^2 x\cdot\cos^2 x\,dx=\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx=\frac{1}{4}\int(1-\cos^2 2x)dx$$

$$=\frac{1}{4}\int dx-\frac{1}{4}\int \cos^2 2x\,dx=\frac{1}{4}x-\frac{1}{4}\int\frac{1+\cos 4x}{2}dx=\frac{1}{4}x-\frac{1}{8}\int(1+\cos 4x)dx$$

$$=\frac{1}{4}x-\frac{1}{8}\int dx-\frac{1}{8}\int \cos 4x\,dx=\frac{1}{4}x-\frac{1}{8}x-\frac{1}{8}\cdot\frac{1}{4}\sin 4x+C=$$

$$=\frac{1}{8}x-\frac{1}{32}\sin 4x+C$$

8. ✓ $\int \sin^6 x\,dx$

11. ✓ $\int \sin^4 2x\,dx$

Rj. $\frac{3}{8}x-\frac{1}{8}\sin 4x+\frac{1}{64}\sin 8x+C$

9. ✓ $\int \sin^2 x\cdot\cos^4 x\,dx$

10. ✓ $\int \cos^6 x\,dx$

Rj. $\frac{5}{16}x-\frac{1}{48}\sin^3 2x+\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x+C$

$$\int \sin\alpha x \cdot \sin\beta x \, dx, \quad \int \sin\alpha x \cdot \cos\beta x \, dx, \quad \int \cos\alpha x \cdot \cos\beta x \, dx$$

koristimo formule:

$$\sin\alpha x \cdot \sin\beta x = \frac{1}{2}\left[\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\right]$$

$$\sin\alpha x \cdot \cos\beta x = \frac{1}{2}\left[\sin(\alpha+\beta)x + \sin(\alpha-\beta)x\right]$$

$$\cos\alpha x \cdot \cos\beta x = \frac{1}{2}\left[\cos(\alpha+\beta)x + \cos(\alpha-\beta)x\right]$$

12. $$\int \sin 4x \cdot \sin 2x \, dx = \frac{1}{2}\int (\cos 2x - \cos 6x)\, dx = \frac{1}{2}\int \cos 2x \, dx -$$
$$- \frac{1}{2}\int \cos 6x \, dx = \frac{1}{2}\cdot\frac{1}{2}\sin 2x - \frac{1}{2}\cdot\frac{1}{6}\sin 6x + C = \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + C$$

13. $$\int \sin 3x \cdot \cos 5x \, dx = \frac{1}{2}\int (\sin 8x + \sin(-2x))\, dx = \frac{1}{2}\int \sin 8x \, dx -$$
$$- \frac{1}{2}\int \sin 2x \, dx = \frac{1}{2}\cdot\left(-\frac{1}{8}\right)\cos 8x - \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cos 2x + C = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C$$

14. $$\int \cos x \cdot \cos 3x \cdot \cos 5x \, dx = \int \frac{1}{2}(\cos 4x + \cos(-2x))\cos 5x \, dx =$$
$$= \frac{1}{2}\int (\cos 4x + \cos 2x)\cos 5x \, dx = \frac{1}{2}\int \cos 4x \cos 5x \, dx + \frac{1}{2}\int \cos 2x \cos 5x \, dx$$
$$= \frac{1}{2}\cdot\frac{1}{2}\int (\cos 9x + \cos x)\, dx + \frac{1}{2}\cdot\frac{1}{2}\int (\cos 7x + \cos 3x)\, dx =$$
$$= \frac{1}{4}\int \cos 9x \, dx + \frac{1}{4}\int \cos x \, dx + \frac{1}{4}\int \cos 7x \, dx + \frac{1}{4}\int \cos 3x \, dx =$$
$$= \frac{1}{4}\cdot\frac{1}{9}\sin 9x + \frac{1}{4}\sin x + \frac{1}{4}\cdot\frac{1}{7}\sin 7x + \frac{1}{4}\cdot\frac{1}{3}\sin 3x + C$$
$$= \frac{1}{36}\sin 9x + \frac{1}{4}\sin x + \frac{1}{28}\sin 7x + \frac{1}{12}\sin 3x + C$$

15. $$\int \sin x \cdot \sin 2x \cdot \sin 4x \, dx \quad \text{Rj. } -\frac{1}{20}\cos 5x - \frac{1}{12}\cos 3x + \frac{1}{28}\cos 7x +$$
$$+ \frac{1}{4}\cos x + C$$

16.$^{\vee}$ $\int \sin 2x \cdot \cos 3x \cdot \sin 5x \, dx$

17. $\int \sin^2 \frac{x}{2} \cos 3x \, dx = \int \frac{1-\cos 2\cdot\frac{x}{2}}{2} \cos 3x \, dx = \frac{1}{2}\int (1-\cos x)\cdot \cos 3x \, dx$

$= \frac{1}{2}\int (\cos 3x - \cos x \cdot \cos 3x) \, dx = \frac{1}{2}\int \cos 3x \, dx - \frac{1}{2}\int \cos x \cos 3x \, dx$

$= \frac{1}{2}\cdot\frac{1}{3}\sin 3x - \frac{1}{2}\cdot\frac{1}{2}\int (\cos 4x + \cos 2x) \, dx = \frac{1}{6}\sin 3x - \frac{1}{4}\int \cos 4x \, dx$

$- \frac{1}{4}\int \cos 2x \, dx = \frac{1}{6}\sin 3x - \frac{1}{16}\sin 4x - \frac{1}{8}\sin 2x + C$

18.$^{\vee}$ $\int \sin^2 2x \cdot \cos^2 3x \, dx$

19.$^{\vee}$ $\int \sin^3 x \cdot \cos^2 2x \, dx$

\# Izračunati integral $I=\int \frac{8\cos x-\sin x}{2\cos x+\sin x}\,dx$.

Rj.

$$(2\cos x+\sin x)'=-2\sin x+\cos x$$

$$\frac{8\cos x-\sin x}{2\cos x+\sin x}=A+B\,\frac{-2\sin x+\cos x}{2\cos x+\sin x}\quad /\cdot(2\cos x+\sin x)$$

$$8\cos x-\sin x=A(2\cos x+\sin x)+B(-2\sin x+\cos x)$$

$$8\cos x-\sin x=(2A+B)\cos x+(A-2B)\sin x$$

$$2A+B=8$$
$$A-2B=-1\quad /\cdot 2$$

$$2A+B=8$$
$$-\;2A-4B=-2$$

$$5B=10$$
$$B=2$$

$$2A+B=8$$
$$2A=8-2$$
$$2A=6$$
$$A=3$$

$$I=\int\frac{8\cos x-\sin x}{2\cos x+\sin x}\,dx=\int\left(3+2\,\frac{-2\sin x+\cos x}{2\cos x+\sin x}\right)dx=$$

$$=3\int dx+2\int\frac{-2\sin x+\cos x}{2\cos x+\sin x}\,dx=\left|\begin{array}{l}2\cos x+\sin x=t\\ (-2\sin x+\cos x)\,dx=dt\end{array}\right|$$

$$=3x+2\int\frac{dt}{t}=3x+2\ln|t|+C=$$

$$=3x+2\ln|2\cos x+\sin x|+C$$